COMPUTING MATRIX REPRESENTATIONS

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ABSTRACT. Let G be a finite group and χ be a faithful irreducible character for G. Earlier papers by the first author describe techniques for computing a matrix representation for G which affords χ whenever the degree $\chi(1)$ is less than 32. In the present paper we introduce a new, fast method which can be applied in the important case where G is perfect and the socle $soc(G/Z(G))$ of G over its centre is abelian. In particular, this enables us to extend the general construction of representations to all cases where $\chi(1) \leq 100$. The improved algorithms have been implemented in the new version 3.0.1 of the GAP package REPSN by the first author.

1. INTRODUCTION

In a series of earlier papers the first author has described how four basic techniques can be used together to compute a representation of a finite group afforded by a specified ordinary character, and shown that these techniques are enough to compute a representation for any character of degree < 32 for any group (see [4], [6]). These methods are quite practical and a package REPSN [5] of programs written in GAP [9] and based on this work is available.

The current paper analyzes further the case of perfect groups which are not central extensions of simple groups. It describes the situations where the known reduction steps fail, and provides new methods for constructing representations in this case when the character degree is < 128. This enables us to extend the general construction of representations to the case where the character has a degree lying in the range from 32 to 100.

Let G be a finite group and χ be a character of degree d. We want to compute a representation of G which affords χ (without loss in generality we shall assume χ is faithful and irreducible). The four basic techniques which we use in computing representations are the following.

R1 [Induction] If there exists a proper subgroup H of G and a character ψ of H such that $\chi = \psi^G$, then we can obtain a representation affording χ by inducing a representation of H afforded by ψ to G.

R2 [Extension] If there exists a proper subgroup H of G such that the restriction χ_H is irreducible then a representation for G affording χ can be computed from a representation of H affording χ_H by solving systems of linear equations in d^2 unknowns (see [6] and also Lemma 6 below).

R3 [Central product] If Z is a subgroup of the centre of G, and $G = HK$ for two proper normal subgroups H and K with $H \cap K = Z$, then there exist irreducible characters ψ and φ of H and K, respectively, such that

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$\chi(xy) = \psi(x)\varphi(y)$ for all $x \in H$ and $y \in K$. Moreover a representation affording χ can be constructed from representations affording ψ and φ, respectively, using a tensor product (see [6]).

R4 [χ-subgroup] If there exists a subgroup H of G such that the restriction χ_H has a constituent α of degree 1 and multiplicity 1, then a method of computing a representation affording χ is presented in [8]. (A subgroup H with this property is called a χ-*subgroup*. See [6].)

It is shown in [6] that if G is solvable, then R1 and R2, applied recursively, are sufficient to compute a representation afforded by any specified character. More generally it is shown that these two techniques can be used recursively to reduce to the case where G is perfect. Thus the general problem is reduced to the case where G is perfect.

If G has a normal subgroup N such that $\chi_N = e(\theta_1 + \cdots + \theta_s)$ where $\theta_1, \ldots, \theta_s$ are distinct irreducible constituents with $e \geq 1$ and $s > 1$, then χ is imprimitive. Specifically, the inertia group $T := I_G(\theta_1)$ has index s in G, and there exists an irreducible constituent ψ of χ_T of degree $\chi(1)/s$ so $\chi = \psi^G$ (see [11, Theorem 6.11]). Thus in this case we can use R1 to reduce the problem to computation of a representation affording the character ψ which has smaller degree. We shall therefore consider the case where for each $N \lhd G$ the restriction χ_N is a multiple of a single irreducible character. In particular, because we are assuming that χ is faithful, this implies that the centre $Z(G)$ is the maximal normal abelian subgroup of G, and indeed cyclic since χ is irreducible.

2. Simple groups and their central covers

The case where G is a simple group or a central cover of a simple group and χ is an irreducible character of degree ≤ 100 has been dealt with by the first author in [7]. He has shown that for almost all such pairs (G, χ), G has a specified subgroup H which is a χ-subgroup and hence R4 can be applied to compute the representation. For two groups (the double covers $2.A_{12}$ and $2.A_{13}$) for which there are characters χ of degree 32 with no χ-subgroups, he provides a maximal subgroup M such that the restriction χ_M is irreducible and so R2 can be applied as a recursive step. The only characters of degree ≤ 100 for which neither of these two techniques applies are two (algebraically conjugate) irreducible characters of degree 36 of $6.A_7$. An earlier version of the current paper pointed out that the construction of representations of $6.A_7$ affording these characters was an open problem.

An anonymous referee of the present paper has solved this problem. Briefly, the solution is as follows. The projective unitary group $U_4(3)$ contains a maximal subgroup isomorphic to A_7 (indeed it has four conjugacy classes of these subgroups which are fused under the automorphism group of $U_4(3)$), and $U_4(3)$ has a 12-fold projective cover H of the form $12_2.U_4(3)$ (see [2, p. 52]). Each of the A_7 subgroups lifts in H to a central product of a cyclic group of order 4 with a group of type $6.A_7$, and so H has a subgroup G of type $6.A_7$. The irreducible characters of degree 36 for H restrict to irreducible characters of G. There are now three steps: (i) a faithful character χ of degree 36 is computed for H; (ii) REPSN is used to compute a representation R of H affording χ; and (iii) R is restricted to G to obtain the required representation.

In what follows we show how to reduce the more general problem (where G is perfect) to the special case where G is simple or the central cover of a simple group.

3. Use of projective representations

As well as the techniques R1-R4 described above we need one further technique based on the following result of Schur (see [3, Theorem 51.7]).

Lemma 1. *Let χ be an irreducible character of a finite group G and N be a normal subgroup of G such that $\chi_N = e\theta$ for some $\theta \in Irr(N)$ and integer $e > 1$. Then there exist irreducible projective representations U and V of G/N and G, respectively, such that:*

(a) V is of degree $\theta(1)$ and its restriction to N is an ordinary representation affording θ;

(b) U has degree e;

(c) $x \mapsto U(Nx) \otimes V(x)$ is an ordinary representation of G affording χ.

Recall that $V : G \to \mathrm{GL}(m, \mathbb{C})$ is a projective representation if there exists a nonvanishing function $\lambda : G \times G \to \mathbb{C}$ such that $V(xy) = V(x)V(y)\lambda(x, y)$ for all $x, y \in G$. Schur proved that for every finite group G there exists a finite group H (now called a *Schur cover* of G) and a surjective homomorphism $\varphi : H \to G$ with $K := \ker \varphi \leq Z(H) \cap H'$ such that every projective representation V of G is related to some ordinary representation S of H via the relation $V(\varphi(x)) = S(x)$ for all $x \in H$. The kernel K is uniquely determined up to isomorphism by G and is called the *Schur multiplier* (denoted by $M(G)$). If G is perfect, then the Schur cover H is determined up to isomorphism, although there may be nonisomorphic Schur covers when G is not perfect. For many simple groups, $M(G)$ is quite small, often of order 1 or 2; see [2] (especially pp. xvi and xxii) for more information, including character tables for some covering groups of simple groups.

For our purposes we restrict ourselves to the case where G is perfect and apply the lemma in cases where we know enough about the covering group H for G/N to be able to compute representations affording each of the irreducible characters of degree e for H. Hence we assume that we can compute candidate projective representations U of G/N (the number of such representations is equal to the number of irreducible characters of H of degree e). To compute V we proceed as follows.

Let T be a set of generators for N and compute $V_N(t)$ $(t \in T)$ such that $t \mapsto V_N(t)$ determines a representation V_N of N affording θ. Put $V(x) := V_N(x)$ for all $x \in N$. For each $x \in G \smallsetminus N$ consider the matrices A which satisfy the linear equations

$$AV_N(x^{-1}tx) = V_N(t)A \text{ for all } t \in T.$$

In the case which we shall consider below, N is nilpotent and so we can choose V_N to be monomial and then these equations are sparse. Because θ is G-invariant, the representation $y \mapsto V_N(x^{-1}yx)$ for N is equivalent to V_N and so Schur's lemma shows that there exists an invertible matrix A satisfying these equations and that any other solution is a scalar multiple of A. We choose one such solution A and put $V(x) := A$. (Section 6 describes a method of computing suitable $V(x)$ in the case in which we are interested which does not require solution of linear equations.) It is now straightforward to verify that $x \mapsto V(x)$ is a projective representation of G and that for one of the candidate projective representations U and for some function $\mu : G \to \mathbb{C}$, $x \mapsto U(Nx) \otimes V(x)\mu(x)$ is an (ordinary) representation of G affording χ (compare with the proof of [3, Theorem 51.7]). Note that $\mu(x)$ is determined completely for those $x \in G$ for which $\chi(x) \neq 0$ since $\chi(x) = (tr\ U(Nx))(tr\ V(x))\mu(x)$.

If there is more than one candidate U then it is necessary to check that with these values of $\mu(x)$ we actually have a representation of G affording χ. If there are many such candidates then, rather than compute many projective representations U, we can use an approach where we can eliminate various candidates simply from their character values. Let H be a projective cover of G/N with the covering homomorphism $\varphi : H \rightarrow G/N$ and fix a mapping $x \mapsto \tilde{x}$ of G into H which is constant on each coset of N and such that $\varphi(\tilde{x}) = Nx$. Then for each projective representation U of G/N there is an (ordinary) representation $\tilde{U}$ of H for which $U(Nx) = \tilde{U}(\tilde{x})\lambda(Nx)$ for all $Nx \in G/N$ and some nonzero $\lambda(Nx) \in \mathbb{C}$. Now if the character χ is afforded by the representation R of G where $R(x) := U(Nx) \otimes V(x)\mu(x)$ as above, then $R(x) = \tilde{U}(\tilde{x}) \otimes V(x)\nu(x)$ where $\nu(x) := \mu(x)\lambda(Nx)$. Let $\{x_1, \ldots, x_k\}$ be a generating set for G and set $A_i = V(x_i)$ for each i. Then for each word $w(X_1, \ldots, X_k)$ in the commutator subgroup of the free group on $\{X_1, \ldots, X_k\}$ we have
$$R(w(x_1, \ldots, x_k)) = \tilde{U}(w(\tilde{x}_1, \ldots, \tilde{x}_k)) \otimes w(A_1, \ldots, A_k)$$
since R and $\tilde{U}$ are representations and the sum of the exponents of powers of X_i in w is 0 for each i. Taking traces, this shows that
$$\chi(w(x_1, \ldots, x_k)) = \eta(w(\tilde{x}_1, \ldots, \tilde{x}_k)) \; tr \; w(A_1, \ldots, A_k)$$
where η is the character of the ordinary representation $\tilde{U}$ of H. Since H is perfect, every element of H has the form $w(\tilde{x}_1, \ldots, \tilde{x}_k)$ for some commutator word w and we can compute $\eta(w(\tilde{x}_1, \ldots, \tilde{x}_k))$ whenever $tr \; w(A_1, \ldots, A_k) \neq 0$.

An especially simple case occurs when both U and V are ordinary representations. We have the following criterion for this to happen [6, Theorem 4.3.7] (the result is proved under additional hypotheses, but these are easily seen to be unnecessary).

Lemma 2. *The representations U and V in Lemma 1 may be chosen as ordinary representations if there exists $\psi \in Irr(G)$ such that $\psi_N = \theta$ (briefly, θ is extendible to G). If e is a prime then either θ is extendible to G or χ_P is irreducible for each Sylow e-subgroup P/N of G/N.*

4. First reduction for perfect groups

As noted in the Introduction, by using R1 and R2 recursively we can assume that G is perfect and χ_N is a multiple of a single irreducible constituent for each $N \lhd G$.

We shall use the following lemma (see [6, Theorem 3.2.3]).

Lemma 3. *Let G be a finite group with centre Z and consider the socle S/Z of G/Z. Then we can write $S/Z = T_0/Z \times T_1/Z \times \cdots \times T_n/Z$ where T_0/Z is abelian and the other T_i/Z are nonabelian simple groups (possibly T_0/Z is trivial or $n = 0$). Let C be the centralizer in G of the nonabelian part of S/Z. Then*

(i) SC is a central product of $C, T_1, \ldots, T_n$. Furthermore if T_0/Z is trivial, then C/Z is trivial.

(ii) There exists a normal subgroup K of G with $G \geq K \geq SC$ such that G/K acts faithfully by conjugation on the set $\{T_1/Z, \ldots, T_n/Z\}$, and K/SC is solvable.

(iii) If G is perfect and $n \leq 4$ then $G = SC$.

First suppose that G is perfect but $n > 4$. Then in the range in which we are interested we have the following result.

Corollary 4. *Under the conditions of the lemma suppose that G is perfect and that $n > 4$. Assume that G has a faithful irreducible character χ of degree < 128 and that $\chi_S = e\theta$ for some $\theta \in Irr(S)$. Then one of the following holds.*

(a) $G = SC$ so use of R3 allows us to reduce our problem to smaller groups.

(b) $n = 5$ and $e = 1, 2$ or 3. Either χ_P is irreducible when P/S is a Sylow e-subgroup of G/S (take $P = S$ for $e = 1$) and we can apply R2, or $e > 1$ and there exists an extension ψ of θ to G. In the latter case χ is afforded by a representation of the form $x \mapsto U(Sx) \otimes V(x)$ where U and V are ordinary representations of G/S and G, respectively, and V affords ψ.

Proof. Since χ is faithful, the character θ is also faithful, and $e\theta(1) < 128$. Since S is a central product of $T_0, T_1, \ldots, T_n$, $\theta(x) = \varphi_0(x)\varphi_1(x)\ldots\varphi_n(x)$ for all $x \in S$ where φ_i is an irreducible character of T_i with $Ker\varphi_i \leq Z$. Since T_i is nonabelian for $i > 0$, we have $\varphi_i(1) \geq 2$ for $i > 0$ and so $\theta(1) \geq 2^n$ (the only cover of a simple group with a faithful irreducible character of degree 2 is $2.A_5$). Since $n > 4$ by hypothesis, we must have $n = 5$, and $e = 1, 2$ or 3. Because G is perfect, it follows from (ii) that $G = SC$ unless the T_i/Z $(i = 1, \ldots, 5)$ are isomorphic and $G/K \cong A_5$. The remaining claims follow from Lemma 2. $\square$

On the other hand suppose that G is perfect and $n \leq 4$. Then Lemma 3 shows that one of the following is true:

(a) G is a central product of two or more groups and so R3 can be used to reduce our problem to a smaller group;

(b) $G = S = T_1$ is a central cover of a nonabelian simple group; or

(c) $S/Z = T_0/Z$ and the socle of G/Z is abelian

Case (b) has already been considered (see Section 2), so only case (c) remains. This is dealt with in the next section.

5. Perfect groups with $soc(G/Z)$ abelian

Consider now a perfect group G with an irreducible character χ such that χ_N is a multiple of a single irreducible constituent for each $N \lhd G$. In particular, $Z := Z(G)$ is the unique maximal normal abelian subgroup. We now assume that $S/Z := soc(G/Z)$ is abelian.

We shall need the following theorem of D.A. Suprunenko (see [6, Section 4.3]).

Lemma 5. *Let G be any group with a faithful irreducible character χ. Suppose that $Z := Z(G)$ is the unique maximal normal abelian subgroup of G, and that the Fitting subgroup $F := Fit(G)$ properly contains Z. Then*

(a) F/Z is the unique maximal normal abelian subgroup of G/Z.

(b) $\chi_F = e\theta$ with $\theta \in Irr(F)$ and $|F : Z| = m^2$ where $m := \theta(1)$.

(c) The Sylow subgroups of F/Z are elementary abelian.

(d) Suppose that $m = p_1^{l_1} \ldots p_s^{l_s}$ is the prime decomposition of m. Then there exists a homomorphism

$$G \to \prod_{i=1}^{s} Sp(2l_i, p_i)$$

with kernel $C_G(F/Z)$.

(e) If $soc(G/Z)$ is abelian (and hence equal to F/Z by (a) and (c)), then $C_G(F/Z) = F$.

We apply this to our perfect group G with the assumption that $\chi(1) \leq 100$. Since $soc(G/Z)$ is abelian, $F \neq Z$ and so all the conclusions of Lemma 5 hold. If

χ_F is irreducible ($e = 1$), then we can use R2 to reduce the problem to a group of smaller order, and if e is prime then Lemma 2 may be applied. Thus we may suppose that $e \geq 4$. In particular, $m \leq 25$. Since $Sp(2,2)$ and $Sp(2,3)$ are solvable, and G/S can be embedded into $\prod_{i=1}^{s} Sp(2l_i, p_i)$ by (d) and (e), we conclude that G/S is isomorphic to a perfect subgroup of one of the following groups:

(i) ($m = p$) $Sp(2,p)$ ($= \mathrm{SL}(2,p)$) for $p = 5, 7, 11, 13, 17, 19$ or 23;

(ii) ($m = 2^2, 2^3, 2^4$ or 3^2) $Sp(4,2)$, $Sp(6,2)$, $Sp(8,2)$ and $Sp(4,3)$, respectively;

or

(iii) ($m = 2^2 5$ or 5^2) $Sp(4,2) \times Sp(2,5)$ or $Sp(4,5)$, respectively.

In GAP there is a program due to D. F. Holt which can be used to construct a Schur cover of G/S (`SchurCover` and `EpimorphismSchurCover`) provided the group is not too large. The projective representation method described in Section 3 can then be used to construct a representation of G afforded by χ provided we can compute the values of $V(x)$ referred to there. We consider an efficient way to compute these $V(x)$ in the next section.

6. Extending projective representations

In his paper [12] Minkwitz gives a method of extending an ordinary representation from a subgroup H on which it is irreducible to the full group G (see also [1]). Unlike the method proposed in [6] this does not involve solution of a system of equations. We shall generalize Minkwitz's method to construct the projective representation V described in Section 3 when G is perfect and satisfies the hypotheses of Lemma 5.

In general, let H be a subgroup of any group G and suppose $\chi \in Irr(G)$ has degree d and $\chi_H = e\theta$ for some integer $e \geq 1$ and some $\theta \in Irr(H)$. Thus θ has degree d/e. Let S be a representation of H affording θ and R be a representation of G affording χ for which $R(u) = I_e \otimes S(u)$ for all $u \in H$.

Lemma 6. *Put $\gamma := d/(|H|\, e^2)$, and for each $x \in G$ define (following Minkwitz):*

$$V(x) := \gamma \sum_{u \in H} \chi(xu^{-1}) S(u).$$

Then:

(a) $V(1) = I$.

(b) $V(xv) = V(x)S(v)$ and $V(vx) = S(v)V(x)$ for all $v \in H$ and $x \in G$ and in particular $V(v) = S(v)$ for all $v \in H$ by (a).

(c) $\mathrm{tr}\, V(x) = (1/e)\chi(x)$ for all $x \in G$.

(d)[Minkwitz] If $e = 1$, then $V(x) = R(x)$ for all $x \in G$.

(e) A necessary and sufficient condition for $V(x) \neq 0$ is that there exists $u \in H$ such that $\chi(xu) \neq 0$.

(f) If $H \lhd G$, then whenever $V(x) \neq 0$ we have $V(x)^{-1}S(v)V(x) = S(x^{-1}vx)$ for all $x \in G$ and $v \in H$.

(g) If Z is a subgroup of the centre of G with $Z \leq H$, and T is a set of right coset representatives for Z in H, then

$$V(x) = \frac{d}{e^2\, |T|} \sum_{u \in T} \chi(xu^{-1}) S(u)$$

(which may speed up the computation of $V(x)$).

Proof. (a) Since χ is a class function, $V(1)$ is invariant under conjugation by each $S(v)$ ($v \in H$). Therefore $V(1) = \lambda I$ for some scalar λ by Schur's lemma. Now since the character inner product $[\chi_H, \theta] = e$ by hypothesis and S has degree d/e:

$$(d/e)\lambda = tr\ V(1) = \gamma \sum_{u \in H} \chi(u^{-1})\theta(u) = (d/e^2)[\chi_H, \theta] = (d/e)$$

and so $V(1) = I$.

(b) Since S is a representation

$$V(xv) = \gamma \sum_{u \in H} \chi(xvu^{-1})S(u) = \gamma \sum_{w \in H} \chi(xw^{-1})S(w)S(v) = V(x)S(v)$$

and the proof of the second equality is similar.

(c) For $u \in H$ we have $R(u) = I_e \otimes S(u)$ and so

$$tr\ V(x) = \gamma \sum_{u \in H} \chi(xu^{-1})\theta(u) = tr\ \left\{ \gamma \sum_{u \in H} \theta(u)R(x)R(u^{-1}) \right\}$$

$$= tr\ \left\{ R(x)\left(I_e \otimes \gamma \sum_{u \in H} \theta(u^{-1})S(u) \right) \right\} = (1/e)tr\ R(x)$$

by (a) since $\chi(u^{-1}) = e\theta(u^{-1})$.

(d) Since S is irreducible, a theorem of Burnside (see, for example, [3, Theorem 27.4] shows that $S(u)$ ($u \in H$) spans the full $(d/e) \times (d/e)$ matrix algebra. Since $e = 1$, this shows that for each $x \in G$ we can write $R(x) = \sum_{v \in H} \xi_v S(v)$ for some scalars ξ_v. Thus using (a) we obtain

$$R(x) = \sum_{v \in H} \xi_v V(v) = \gamma \sum_{u \in H} S(u) \sum_{v \in H} \xi_v \chi(vu^{-1}).$$

However

$$\sum_{v \in H} \xi_v \chi(vu^{-1}) = tr\ \left(\sum_{v \in H} \xi_v R(v)R(u^{-1}) \right) = tr\ R(x)R(u^{-1}) = \chi(xu^{-1})$$

and so $R(x) = \gamma \sum_{u \in H} \chi(xu^{-1})S(u) = V(x)$ as claimed.

(e) The condition is necessary from the definition of $V(x)$. Conversely, suppose that $\chi(xu) \neq 0$ for some $u \in H$. Then $V(xu)$ is nonzero by (c) and so (b) implies that $V(x) \neq 0$. Thus the condition is also sufficient.

(f) Since S and the representation $v \mapsto S(x^{-1}vx)$ of H are irreducible, Schur's lemma shows that every nonzero $(d/e) \times (d/e)$ matrix C with the property $S(v)C = CS(x^{-1}vx)$ for all $v \in H$ is invertible. By (b) the matrix $C = V(x)$ has this property.

(g) Let $z \in Z$. Since R is irreducible, $R(z) = \zeta I$ for some scalar ζ. Then for each $u \in T$ and $x \in G$ we have $R(x(zu)^{-1}) = \zeta^{-1}R(xu^{-1})$ and $S(zu) = \zeta S(u)$. Thus $\chi(x(zu)^{-1})S(zu) = \chi(xu^{-1})S(u)$ for each $z \in Z$ and the result follows. $\square$

We want to use this theorem to construct the projective representation V described in Section 3 for our perfect group G where N is equal to the Fitting subgroup F. Lemma 6(f) shows that when $V(x) \neq 0$ we can take the matrix $V(x)$ as the value for such a projective representation at x. To ensure that we can obtain the projective representation at all elements of G it is necessary to show that $L := \langle x \in G \mid V(x) \neq 0 \rangle$ is equal to G.

Suppose the contrary. By part (e) we see that $V(x) = 0$ implies that $V(y) = 0$ for every conjugate of x, and so L is a normal subgroup. Since $L \neq G$ by assumption, we can find a maximal normal subgroup K of G with $L \leq K$; since G is perfect, G/K is a nonabelian simple group. On the other hand $\chi(x) = 0$ for all $x \notin K$ by part (e) of the lemma above and so Lemma 2.29 of [11] shows that $[\chi_K, \chi_K] = |G/K|$. By the hypothesis on χ, $\chi_K = f\varphi$ for some integer f and some $\varphi \in Irr(K)$, and so $[\chi_K, \chi_K] = f^2$ where $f^2 \leq \chi(1)^2 \leq 10^4$. However, the list of orders of simple groups given in [2] shows that no simple group has an order $\leq 10^6$ which is a square (the order of $Sp(4, 41)$ is a square, but its size is approximately 6.7×10^{15}). This contradiction shows that $L = G$, and so we can find enough elements x with $V(x) \neq 0$ to generate the whole of the projective representation.

Remark 7. *Minkwitz introduced his extension theorem (part (d) of the lemma) as a means of computing an irreducible representation of G when the restriction to a subgroup H is irreducible and known. Whether it is faster than solving of a system of linear equations as described in [6] depends on a number of factors. When H is small, Minkwitz's method is much faster, but the advantage of the method described in [6] is that the complexity of the computation depends only on the degree of the representation (an $O(d^6)$ computation using standard methods of solving linear equations), but not on the size of H. Generally speaking, Minkwitz's method tends to be faster when the size of the subgroup H is not too large, but is slower than the method described in [6] when we have to deal with larger subgroups. In the specific case in which we apply the generalized Minkwitz method to construct projective representations, H is the Fitting subgroup F of G, and Lemma 5 applies. Thus $|F : Z| = (d/e)^2$ and the computation in part (g) of the lemma is very fast.*

Remark 8. *We also examined an alternative approach when G is a perfect group with $soc(G/Z)$ abelian (and the hypotheses of Lemma 5 hold); namely, to try to find a χ-subgroup and use R4. The GAP library of finite perfect groups provides, up to isomorphism, a list of all 1096 perfect groups whose sizes are less than 10^6 excluding the sizes $61440, 86016, 122880, 172032, 245760, 344064, 368640, 491520, 688128, 737280$ and 983040 (it is based on [10]). The limitation on the size of the groups means that not all perfect groups which have a character of degree at most 100 are included. For each pair (G, χ) where G is a group from this GAP library and χ is an irreducible character of degree at most 100 we found a suitable χ-subgroup by making a direct search.*

However, computing a representation affording χ using a χ-subgroup turns out to be generally slower than using the method given above. Thus it does not seem worthwhile to provide the tables of χ-subgroups needed for this alternative approach.

7. RUNTIMES

The algorithms described in this paper for computing representations of perfect groups G for which $soc(G/Z(G))$ is abelian have been implemented in GAP and are incorporated in a new version of the package REPSN [5]. In the following table we give the time spent by GAP to construct representations of seven different perfect groups affording a faithful irreducible character χ. These are perfect groups G, available in the GAP library, such that $Z(G)$ is a maximal normal abelian subgroup of G, $F/Z(G) := soc(G/Z(G))$ is abelian and $\chi_F = e\theta$ for some $\theta \in Irr(F)$. The table gives the cpu time (processor time) in seconds (the machine used was an Apple

G5 with dual 2.5 GHz processors). The notation in the first column describes the group in accordance with [10].

G	$|G|$	$\chi(1)$	time
A_5 $2^{4'}$ **C** **N** 2^1	1920	16	4
A_5 $(2^4$ **E** $(2^1$ **A** $\times 2^1))$ **C** 2^1	7680	24	14
A_5 2^1 5^2 **C** 5^1	15000	30	16
A_6 $(2^4$ **E** 2^1 **A**) **C** 2^1	23040	32	44
$L_3(2)$ 2^1 7^2 **C** 7^1	115248	56	68
$L_2(8)$ **N** $(2^6$ **E** 2^1 **A**) **C** 2^1	129024	64	112
A_7 $2^1 \times (2^6$ **C** $2^1)$	645120	48	690

Table 1: EXAMPLES OF RUNTIMES

ACKNOWLEDGEMENT

REFERENCES

[1] M. Clausen, *A direct proof of Minkwitz's extension theorem*, Appl. Algebra Engrg. Comm. Comput., **8** (1997) 305–306.
[2] J.H. Conway et al, *Atlas of Finite Groups*, Clarendon Press, Oxford, 1985.
[3] C.W. Curtis and I. Reiner, *Representation Theory of Finite Groups and Associative Algebras*, Interscience-Wiley, 1962.
[4] V. Dabbaghian-Abdoly, *An Algorithm to Construct Representations of Finite Groups*, Ph.D. thesis, Carleton University, May 2003.
[5] V. Dabbaghian-Abdoly, *REPSN - for constructing representations of finite groups*, GAP package, Version 3.0.1, (2008). (http://www.gap-system.org/Packages/repsn.html).
[6] V. Dabbaghian-Abdoly, *An algorithm for constructing representations of finite groups*, J. Symbolic Comput., **39** (2005) 671–688.
[7] V. Dabbaghian-Abdoly, *Constructing representations of higher degrees of finite simple groups and covers*, Math. Comp., **76** (2007) 1661–1668.
[8] J.D. Dixon, *Constructing representations of finite groups*, DIMACS Ser. Discrete Math. Theoret. Comput. Sci. **11**, Amer. Math. Soc., Providence, RI (1993), 105–112.
[9] The GAP Group, *GAP—Groups, Algorithms, and Programming*, Version 4.4.12 (2008). (http://www.gap-system.org).
[10] D.F. Holt and W. Plesken, *Perfect Groups*, Clarendon Press. Oxford, 1989.
[11] I.M. Isaacs, *Character Theory of Finite Groups*, Academic Press, New York, 1976.
[12] T. Minkwitz, *Extensions of irreducible representations*, Appl. Algebra Engrg. Comm. Comput., **7** (1996) 391–399.

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